

Bursts in the Chaotic Trajectory Lifetimes Preceding the Controlled Periodic Motion

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(February 8, 2008)

Abstract

The average lifetime ($\tau(H)$) it takes for a randomly started trajectory to land in a small region (H) on a chaotic attractor is studied. $\tau(H)$ is an important issue for controlling chaos. We point out that if the region H is visited by a short periodic orbit, the lifetime $\tau(H)$ strongly deviates from the inverse of the naturally invariant measure contained within that region ($\mu_N(H)^{-1}$). We introduce the formula that relates $\tau(H)/\mu_N(H)^{-1}$ to the expanding eigenvalue of the short periodic orbit visiting H .

05.45.Gg, 05.45.Ac, 05.45.-a

Controlling chaos by stabilizing one of the many unstable periodic orbits embedded within a given chaotic attractor is attainable with small, time-dependent changes in an accessible system parameter [1–3]. The idea is to observe a typical trajectory of the uncontrolled system for some transient time, until it falls sufficiently close to the desired periodic orbit, and then to activate the control mechanism. An important issue related to the utilization of this method is the average lifetime of chaotic transients that precede the controlled periodic motion [1,3–5].

Suppose that the uncontrolled chaotic attractor A describes the asymptotic behavior of the dynamical system $O : D \rightarrow D$, $D \subseteq \mathbf{R}^m$, also referred to as the original system. Let $\vec{\xi} = O^k(\vec{\xi})$ be a point on a particular unstable periodic orbit [6,7] that we wish to stabilize. Furthermore, let the vicinity of the orbit $H \equiv H_\epsilon(\vec{\xi})$ be an m -dimensional ball of radius $\epsilon \ll 1$ centered at $\vec{\xi}$. The probability that a randomly started trajectory does not reach H up to time t is $\sim e^{-t/\tau(H)}$. The average lifetime $\tau \equiv \tau(H)$ is strongly correlated with the visitation frequency of typical trajectories to the region H , which is described in terms of the naturally invariant measure (μ_N) contained within $H - \mu_N(H)$ [1,4,8]. Obviously, if a certain region on a given chaotic attractor is visited more frequently by typical trajectories, the average lifetime it takes for an orbit to land in that region will be smaller. In the present study we address the following question: What is the deviation of τ from $\mu_N(H)^{-1}$ as a function of $\vec{\xi}$ and ϵ ?

We will demonstrate the existence of bursts in the lifetimes, i.e. significant deviations of τ from $\mu_N(H)^{-1}$, which appear when the H region encompasses a point on a short periodic orbit. In contrast to the overall $\tau \simeq \mu_N(H)^{-1}$ behavior, at these exceptional positions, the lifetime τ is considerably prolonged as compared to $\mu_N(H)^{-1}$. As the length of the shortest cycle visiting H increases, the parameter of this deviation, $S(H) \equiv \tau/\mu_N(H)^{-1}$, decreases rapidly towards 1. We will introduce a formula that relates the parameter $S(H)$ to the repelling properties (expanding eigenvalue) of the shortest cycle within H . Furthermore, we will demonstrate that $S(H)$ is independent of ϵ (for $\epsilon \ll 1$). This is consistent with the previously reported scaling $\tau \sim \mu_N(H)^{-1}$ (see e.g. [1,4]).

The present study is motivated by the previous investigation of the logistic map with a hole [9]. In this paper we present a theoretical explanation for the phenomenological result reported in Ref. [9] and generalize it to 1D noninvertible and 2D invertible chaotic maps. (From our considerations a conjecture follows that similar phenomena occur generally in chaotic systems.)

It will be useful to define an auxiliary modified map [9,10]

$$M(\vec{\xi}') = \begin{cases} O(\vec{\xi}'), & \vec{\xi}' \in D \setminus H \\ \text{outside of the basin of } A, & \vec{\xi}' \in H. \end{cases} \quad (1)$$

A typical trajectory of the map O remains on the chaotic attractor forever, while the same trajectory in the map M eventually escapes through the region H , from now on also referred to as the hole. The average lifetime of chaotic transients created by the map M is equal to $\tau(H)$, which we have defined above. Similar maps with a forbidden gap region arise in the context of communicating with chaos [11], and in calculation of the topological entropy [12].

To illustrate the concept of bursts, we consider two chaotic 1D maps: (i) the asymmetric tent map $O(x) = k_1 x$, $x < k_1$; $O(x) = k_2(1 - x)$, $x > k_1$, $k_1, k_2 > 0$, $k_1^{-1} + k_2^{-1} = 1$, and (ii) the sinusoidal map $O(x) = \sin \pi x$. Fig. 1 displays τ as a function of the position of the hole ξ ($H = (\xi - \epsilon, \xi + \epsilon)$), for the two paradigmatic maps (see also Ref. [9]). The width of the hole is kept constant ($\epsilon = 0.005$). Both graphs exhibit some common features: (i) the overall behavior of lifetimes follows the $\mu_N(H)^{-1}$ pattern; (ii) strong local deviations from the $\mu_N(H)^{-1}$ behavior - the bursts, observed as leaps in the lifetimes, occur when the hole interval sweeps across a short periodic orbit; (iii) the bursts are more significant if the length of the short periodic orbit is smaller.

The explanation of the burst phenomenon requires the comparison of two concepts: (i) the conditionally invariant measure [13–15] (also referred to as the c -measure) - the concept associated with the modified system, and (ii) the naturally invariant measure [7,3] of the original system. In order to define these measures, imagine that we cover the chaotic attractor with cells (I) from a very fine grid. Then we randomly distribute a large number (N) of points on the grid, and evolve them under the dynamics O for a long time T . Suppose

that all initial points are colored blue, and that a point irretrievably changes color from blue to red immediately after its first entrance into the region H . Thus, the point $\vec{x}_T = O^T(\vec{x}_0)$ at time T is blue if $O^t(\vec{x}_0) \notin H$ for $t \in \{0, 1, \dots, T-1\}$, and red otherwise. In the limit $T \rightarrow \infty$, the fraction of points found in a given cell I , is just the natural measure contained within that cell: $\mu_N(I) = N_T(I)/N$; $N = \sum_I N_T(I)$, where $N_T(I) = b_T(I) + r_T(I)$ denotes the total number of points in a given cell I at time T [3]. The number of blue (red) points within a given cell I at time T is denoted as $b_T(I)$ ($r_T(I)$). The points which change color from blue to red under the action of the map O , are those which would escape the attractor under the action of the map M . Hence, if we were to evolve exactly the same initial conditions using the map M for the same time T , the number of surviving points (blue points) would be $B_T = \sum_I b_T(I) \sim N \exp(-T/\tau)$. In the limit $N \rightarrow \infty$, $T \rightarrow \infty$, the distribution of blue points converges to the c -measure of the modified system [13–15]. The fraction of blue points in a given cell I is simply the c -measure contained within that cell: $\mu_C(I) = b_T(I)/B_T$.

The blue point at any time $t > 0$, was certainly not in the hole H at time $t-1$. Therefore, $M^{-1}(I) \equiv O^{-1}(I) \setminus H$ and $b_T(I) = b_{T-1}(M^{-1}(I))$, which divided by $B_T = B_{T-1} \exp(-1/\tau)$ yields the well known relation for the c -measure [13–15]

$$\mu_C(I) = e^{\frac{1}{\tau}} \mu_C(M^{-1}(I)). \quad (2)$$

By summing the equation above over all the cells I we obtain

$$\frac{1}{\tau} = -\ln[1 - \mu_C(H)] \simeq \mu_C(H). \quad (3)$$

We emphasize the importance of this observation. The average lifetime it takes for a typical trajectory to reach the small region H on the attractor is an inverse of the c -measure contained within that region ($\mu_C(H)^{-1}$), which may significantly differ from the inverse of the natural measure ($\mu_N(H)^{-1}$).

As an illustration, in Fig. 2 we display the c -measure for the modified version of the map $\sin \pi x$, in comparison to the natural measure of the original map. For the c -measure in Fig. 2 a), the hole has been positioned at an arbitrary point, but not on the short periodic orbit.

In this case, we observe that $\mu_C(H) \simeq \mu_N(H)$, i.e., $\mu_N(H)^{-1}$ is a good approximation for the lifetime. In contrast, in Fig. 2 b) we display μ_C for the modified map $\sin \pi x$, with the hole positioned on the fixed point. We notice that $\mu_C(H)$ strongly deviates from $\mu_N(H)$. This case corresponds to the burst labeled 1 in Fig. 1 b). The overall agreement of the two measures is evident in both Figures 2 a) and b). However, at locations above the first few images of the hole, μ_C takes the shape of a well, with values which are considerably lower than μ_N . When the hole lies on the fixed point (Fig. 2 b)), the wells are just above the hole itself. This results in a pronounced deviation of $\tau = \mu_C(H)^{-1}$ from $\mu_N(H)^{-1}$, which manifests as a burst.

Now we compare the two measures globally. A chaotic repeller is a set of points on the attractor that never visit the hole [14,15,11]. A trajectory that starts close to the repeller, does not escape the attractor for a long time. Therefore, the blue points at a large time T are located along the unstable manifold of the repeller. Their distribution along this manifold defines the c -measure [14]. Thus, the natural measure is constructed from all the points (at time T) on all the unstable manifolds, whereas the c -measure results only from points on parts of these manifolds. The parts which extend from the repeller up to the hole. As we reduce the size of the hole ϵ , the repeller and its unstable manifold grow. Consequently, μ_C gradually approaches μ_N , and for sufficiently small ϵ , the two measures are practically identical. (For $\epsilon = 0$, μ_C becomes μ_N [13–15].)

However, the deviation of the lifetime $\tau = \mu_C(H)^{-1}$ from $\mu_N(H)^{-1}$ depends only on the values of the two measures within the hole, and therefore is a local quantity. In order to make a more accurate comparison of μ_C and μ_N , we introduce the following definitions. Consider a set $P \subset D$ such that $\mu_N(P) > 0$. We define the quantity

$$a(P) = \mu_C(P)/\mu_N(P), \quad (4)$$

which describes the relation between μ_C and μ_N within P . We also define the influence $i(P)$ of the hole on the set P as

$$i(P) = \frac{\mu_N(O^{-l(P)}(P) \cap H)}{\mu_N(O^{-l(P)}(P))}. \quad (5)$$

$l(P)$ denotes the smallest integer for which the section $O^{-l(P)}(P) \cap H$ becomes nonempty. The natural invariant measure within $O^{-l(P)}(P)$ is mapped to P in $l(P)$ iterates. The influence is just a fraction ($0 \leq i(P) \leq 1$) of $\mu_N(P)$ that is mapped from the hole in the last $l(P)$ time steps.

Let $P_\epsilon \equiv P_\epsilon(\vec{x}) \subset D$ be an m -dimensional ball of radius ϵ (the same radius as the hole) centered at \vec{x} . We ask the following question: Given a chaotic attractor and choosing the hole region, what is the behavior of $a(P_\epsilon) = \mu_C(P_\epsilon)/\mu_N(P_\epsilon)$ as the position of P_ϵ on the attractor is changed?

By using Eq. (2) and the identity $\mu_N(P_\epsilon) = \mu_N(O^{-1}(P_\epsilon))$, we can write

$$a(P_\epsilon) = e^{\frac{l}{\tau}} \cdot (1 - i(P_\epsilon)) \cdot a(M^{-l}(P_\epsilon)), \quad (6)$$

where $l \equiv l(P_\epsilon)$ (in what follows, $l \equiv l(P_\epsilon)$).

Concerning the first factor in Eq. (6), note that the average lifetime typically scales like $\tau \sim 1/\epsilon^{D_p(\vec{\xi})}$ ($D_p(\vec{\xi})$ denotes the pointwise dimension at $\vec{\xi}$) [1,4,3], whereas the minimal number of iterates l for which $O^{-l}(P_\epsilon) \cap H \neq \emptyset$ scales like $l \sim \ln(1/\epsilon)$ [4]. Therefore, $\exp(l/\tau) \simeq 1 + l/\tau \simeq 1$.

If the influence $i(P_\epsilon)$ is small, the second factor in Eq. (6) is $\simeq 1$. We argue that $i(P_\epsilon)$, the influence of the hole on the region P_ϵ decreases exponentially with l . For the 2D original map, $O^{-l}(P_\epsilon)$ is a narrow region which is stretched along the stable direction and squeezed along the unstable one [4]. The intersection of $O^{-l}(P_\epsilon)$ with the hole H is roughly a rectangle of length ϵ and width $\epsilon \exp(-\lambda_1 l)$. For the 1D map, $O^{-l}(P_\epsilon) \cap H$ is an interval of width $\sim \epsilon \exp(-\lambda_1 l)$. In both cases, λ_1 denotes the positive Lyapunov exponent obtained for typical initial conditions on the attractor. Since the natural measure is concentrated along the unstable manifolds [7,3], we can relate $\mu_N(O^{-l}(P_\epsilon) \cap H) \sim \exp(-\lambda_1 l)$. Thus, due to the chaoticity of the map O we obtain $i(P_\epsilon) \sim \exp(-\lambda_1 l)$.

Concerning the third factor in Eq. (6), we consider the set $M^{-l}(P_\epsilon)$ and the value $a(M^{-l}(P_\epsilon))$ in dependence of l . For the 2D maps, the set $M^{-l}(P_\epsilon) \equiv O^{-l}(P_\epsilon) \setminus H$ is stretched exponentially fast with increasing l along the stable manifolds, and thus crosses many of the

unstable manifolds that carry both the natural and the c -measure. For the 1D maps, the number of disjoint intervals that make the l -th preimage of P_ϵ grows exponentially with l . Furthermore, they are scattered all over the attractor. Due to the chaoticity of the map O , in both cases $M^{-l}(P_\epsilon)$ becomes more democratic with larger l , in the sense that the value $a(M^{-l}(P_\epsilon))$ reflects the global agreement between the two measures. Thus, insofar as l is not small, $a(M^{-l}(P_\epsilon)) \simeq 1$.

We conclude that for $l \equiv l(P_\epsilon)$ larger than some critical value (call it l_c), all of the three factors in Eq. (6) are $\simeq 1$ and therefore $\mu_C(P_\epsilon) \simeq \mu_N(P_\epsilon)$. This is consistent with the global agreement between the two measures. The critical value l_c depends on the chaoticity of the original map. For example, we may take l_c to be the smallest integer for which $e^{-\lambda_1 l_c} < 0.1$ (e.g. for the $\sin(\pi x)$ map this gives $l_c \sim 3 - 4$). When H maps to P_ϵ in just a few iterates, so that $l < l_c$, a significant difference is observed between $\mu_C(P_\epsilon)$ and $\mu_N(P_\epsilon)$ (this explains the wells in Fig. 2).

Coming back to the average lifetimes, if the hole does not map back to itself in just a few iterates, i.e., if the shortest periodic orbit within H has a period larger than l_c , then $\mu_C(H) \simeq \mu_N(H)$, or simply $\tau \simeq \mu_N(H)^{-1}$. This explains the overall behavior of lifetimes (see Fig. 1). On the other hand, if H encompasses a short periodic orbit (period $\equiv l(H) < l_c$), the two measures differ within the hole. Quantitatively, we substitute $P_\epsilon \rightarrow H$ in Eq. (6) and approximate $e^{l(H)/\tau} \simeq 1$ and $\mu_C(M^{-l(H)}) \simeq \mu_N(M^{-l(H)})$. This results in

$$\tau \simeq (1 - i(H))^{-1} \mu_N(H)^{-1}. \quad (7)$$

Although $l(H)$ is small, the approximation $a(M^{-l(H)}) \simeq 1$ is justified if $l(M^{-l(H)}) > l_c$, or simply, if the period of the second shortest orbit within H exceeds l_c . We have tested relation (7) and consequently the approximations that lead to it in a number of systems. We have compared $\tau(H)$ with $\mu_N(H)^{-1}$ by changing the position of H from "the most exceptional" point, the fixed point, to longer cycles. In Fig. 3 we display a test of Eq. (7) for the the generalized baker's map (see Ref. [3], p. 75, $\lambda_a = 0.35$, $\lambda_b = 0.40$, $\alpha = 0.40$, $\beta = 0.60$), and for the Hénon map (see Ref. [16] $a = 1.4$, $b = 0.3$). Recalling that $i(H)$ decreases exponentially

with $l(H)$, and considering Eq. (7), we see that the parameter $S(H) = (1 - i(H))^{-1}$ decreases rapidly towards 1 with the increase of $l(H)$ (see Fig. 3). Eq. (7) is robust and can be applied for holes of different shapes, as long as $\tau \gg 1$. If (for the 2D maps) we tailor the hole as a rectangle with sides of length ϵ parallel to the stable and unstable manifold segments, and center it on a short periodic orbit, then

$$\tau \simeq (1 - \Lambda_u^{-1})^{-1} \mu_N(H)^{-1}. \quad (8)$$

Λ_u denotes the magnitude of the expanding eigenvalue of that orbit. Eq. (8) also applies to 1D maps. Note that the approximation $i(H) \simeq \Lambda_u^{-1}$ assumes that the natural measure is smooth along the unstable direction within H . We observe that $\tau/\mu_N(H)^{-1}$ is independent of ϵ . This is in accordance with the statement that the lifetime τ scales with ϵ just like $\mu_N(H)^{-1}$ [1,4].

Let us consider an application of Eq. (8). Suppose that we wish to control a chaotic system around an unstable fixed point. In order to obtain the position of the fixed point, its unstable eigenvalue, and other information required for the control, an observation of the free running system is needed [1,2]. From this observation, we can also evaluate the visitation frequency to the ϵ -vicinity of the fixed point, i.e. $\mu_N(H \equiv H_\epsilon(\vec{\xi}))$. ϵ is determined by the maximally allowed deviation of the control parameter from its nominal value [1,2]. The question of interest is how many iterates are needed on the average (τ), before a chaotic trajectory enters the region H , when the control becomes attainable [1]. The prediction given by $\mu_N(H)^{-1}$ is an underestimate, since we are on the fixed point. For example, if the underlying dynamics of the system is the asymmetric tent map (with the same parameters as in Fig. 1), and if $H = (0.62111801 \dots - 0.002, 0.62111801 \dots + 0.002)$, the estimate for the lifetime $\mu_N(H)^{-1}$ gives 250 iterates. On the other hand, the numerically calculated lifetime is $\simeq 627$ iterates, which is more than twice as long. The lifetime obtained from formula (8) is 641 iterates, which is very close to the numerically calculated lifetime. Thus, Eq. (8) can be utilized to easily and accurately obtain τ from an observation of the free running system.

In summary, we have studied the average lifetime (τ) it takes for a randomly started

orbit to land in a small region (H) on a chaotic attractor. That problem was introduced in Ref. [1] as an important issue for controlling chaos. Our main result is that if a low-period unstable periodic orbit visits the region H , then the lifetime τ significantly deviates from the inverse of the natural measure contained within H ($\mu_N(H)^{-1}$). The parameter of this deviation, $\tau/\mu_N(H)^{-1}$, is a function of the expanding eigenvalue of that low-period orbit.

FIGURES

FIG. 1. $\tau(\xi)$ (solid line) and $\mu_N(H)^{-1}$ (dashed line) vs. ξ for a) the asymmetric tent map ($k_1^{-1} = 0.39$, $k_2^{-1} = 0.61$), and b) the $\sin \pi x$ map. $H = (\xi - \epsilon, \xi + \epsilon)$, $\epsilon = 0.005$.

FIG. 2. Figure displays μ_C (solid line) in comparison to μ_N (dashed line) for the $\sin \pi x$ map ($\epsilon = 0.005$). Fig. 2a) shows μ_C for the hole positioned on $\xi = 0.66$. The hole is not visited by a short periodic orbit. Note that $\mu_C(H) \simeq \mu_N(H)$. Fig. 2b) shows μ_C for the hole located on the fixed point. Note that $\mu_C(H) < \mu_N(H)$. In both figures μ_C and μ_N are globally identical, except at the first 3-4 images of the hole, which are plotted underneath the graphs.

FIG. 3. Numerically evaluated parameter $\tau/\mu_N(H)^{-1}$ for the Hénon map (diamonds) and the Baker map (circles) in comparison to $(1 - i(H))^{-1}$ (horizontal bars). The hole of radius $\epsilon = 0.005$ is centered on the shortest periodic orbit (period = $l(H)$) visiting H .

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